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ON THE DYNAMIC STABILITY OF VISCOELASTIC PERFECT COLUMNS†

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Abstract—The dynamic stability of simple supported perfect columns made of a linearly viscoelastic material and subjected to an axial compressive load, P, smaller than the classical Euler elastic buckling load, P_{cr} is examined. The solution to the integro-differential equation is obtained by means of Laplace transforms. In addition, an approximate solution is also derived by adopting the approximation technique introduced in[1]. The results are applied to a simple "three-element model" viscoelastic column.

1. INTRODUCTION

Among stability problems for structures made of viscoelastic materials, the stability of columns is one of the more interesting problems which has been the subject of a large number of investigations during the last three decades. Many different stability criteria, analysis techniques and material laws have been used or suggested (see[2–8]). In addition to the assumptions introduced with respect to the constitutive laws, various approximate solution techniques have also been employed[1, 9–12]. Consequently, results obtained for the same problem by different investigators vary, depending on the criteria and assumptions used.

The classical Euler method may not be applied to the stability of perfect columns made of a linearly viscoelastic material subjected to an axial compressive load smaller than the classical Eular elastic buckling load unless the existence of an adjacent equilibrium position in addition to the initial configuration is proven. A dynamic approach, on the other hand, can be used for the stability analysis of such perfect columns since such an approach does not depend on the equilibrium concept. Therefore, the aim of this paper is to examine the dynamic stability of a perfect column made of a linearly viscoelastic material and to compare the results obtained from an "exact" solution with those derived in both the dynamic and quasi-static case using an approximation technique presented in[1].

Here, the stability of such a perfect column subjected to an axial compression load, P, smaller than the classical Euler elastic buckling load, P_{cr} is examined. The applied load P is assumed to be constant in time. The constitutive equation of the material is given in functional form with the restriction that the column is at rest at times $t < 0$. For the sake of simplicity the temperature dependence is not taken into account in the functional. Furthermore, it is assumed that the material is elastic at $t = 0$ and that the functional characterizing the viscoelastic property of the material is bounded as $t \to \infty$. It is also assumed that the column is disturbed laterally by a sinusoidal velocity field with amplitude v_0 at midspan as soon as the load P is applied.

Using the constitutive equation and the kinematical relations in the equation of motion, the governing equation of the problem is obtained in the form of an integro-differential equation. The solution to this equation is obtained by means of Laplace transforms. Since the denominator of the transformed function is a cubic, an inverse transform is obtained numerically for the case of a "three element model" viscoelastic column, for several load levels (see Fig. 3). Neglecting the inertia force in the equation of motion, a quasi-static solution is also derived. In addition, an approximate solution for both the dynamic and quasistatic case is given by adopting the approximation technique for the constitutive equation introduced in[1]. The results are applied

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to a simple "three-element model" viscoelastic column (see Fig. 4 and Table 1). To assess the accuracy of this approximate method, the results for a three-element model viscoelastic column are compared with those from the "exact" solution to the same problem.

2. STATEMENT OF THE PROBLEM

Consider a simply supported perfect column of length l subjected to an axially applied load P which is constant in time (see Fig. 1). It is assumed that the column is disturbed laterally by a sinusoidal velocity field with amplitude v_o at midspan as soon as the load P is applied. The equation of motion of the column is obtained as

$$\frac{\partial^2 M(x,t)}{\partial x^2} - P\frac{\partial^2 w(x,t)}{\partial x^2} = m\frac{\partial^2 w(x,t)}{\partial t^2} \tag{1}$$

where $w(x, t)$ denotes the lateral deflection, $M(x, t)$ designates the bending moment and m is the mass density of the column per unit length. The boundary and initial conditions are

$$w(o,t) = w(l,t) = 0; \frac{\partial^2 w(o,t)}{\partial x^2} = \frac{\partial^2 w(l,t)}{\partial x^2} = 0$$
$$w(x,o) = 0, \frac{\partial w}{\partial t}(x,o) = v_o \sin\frac{n\pi x}{l}. \tag{2}$$

The constitutive equation of a linearly viscoelastic material for the one-dimensional case may be written in functional form[13, 14] as

$$\sigma(x,t) = G^*[\epsilon(x,t)] \tag{3}$$

where $\sigma(x, t)$ and $\epsilon(x, t)$ denote stress and strain at point x and time t, respectively, and G^* is an integral operator defining the material properties, the exact form of which depends on the basic constitutive assumptions. Assuming the inverse operator, J^*, to exist, we can write

$$(G^*)^{-1} = J^* \text{ or } (J^*)^{-1} = G^* \tag{4}$$

and the strain may be expressed as

$$\epsilon(x,t) = J^*[\sigma(x,t)]. \tag{5}$$

For example, for a temperature dependent viscoelastic material the strain–stress relation, eqn (3), has the form[15]

$$\sigma(x,t) = G[O, T(x,t)]\,\epsilon(x,t)] - \int_o^t \epsilon(x,t)\frac{\partial}{\partial t} G[t-\tau, T(x,s)]\Big|_{s=\tau}^{t} \,d\tau \tag{6}$$

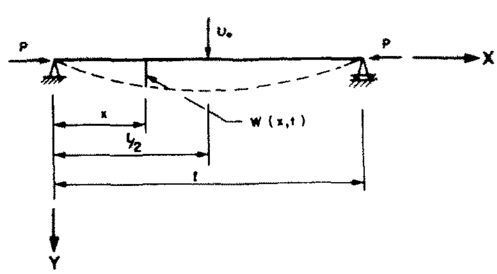

Fig. 1. Loading and geometry of the column.

where $T(x, t)$ denotes the temperature at x and t, and G is a functional of the temperature history, $T(\overset{t}{\underset{s=\tau}{x}}, s)$. For the sake of simplicity, we restrict our treatment in this paper, to temperature-independent viscoelastic materials. Then, eqn (6) is written for a linear viscoelastic solid in the form

$$\sigma(x, t) = E\{\epsilon(x, t) + \int_o^t g(t-\tau)\,\epsilon(x, \tau)\,d\tau\},\ t \geq 0 \tag{7}$$

or, inversely,

$$\epsilon(x, t) = \frac{1}{E}\{\sigma(x, t) + \int_o^t j(t-\tau)\,\sigma(x, \tau)\,d\tau\},\ t \geq 0 \tag{8}$$

where the kernels $g(t-\tau)$ and $j(t-\tau)$, characterizing the viscoelastic property of the material, are functions of time only, and E denotes Young's modulus. Both stress and strain are assumed to vanish for $t < 0$. Further assumptions are:

(i) the initial response of the material is elastic;
(ii) the integrals in (7) and (8) take finite values as $t \to \infty$.

The curvature-bending moment relation for the viscoelastic column can be derived using Bernoulli–Euler beam theory assumptions in the form

$$M(x, t) = IG^*[\kappa(x, t)] \tag{9}$$

where $\kappa(x, t)$ is the time-dependent curvature, while I denotes the second moment of area. Furthermore, the curvature is given in terms of the lateral deflection, $w(x, t)$ as

$$\kappa(x, t) = -\frac{\partial^2 w(x, t)}{\partial x^2} \tag{10}$$

which, with the use of eqns (9) and (10) leads to

$$M(x, t) = -IG^*\left[\frac{\partial^2 w(x, t)}{\partial x^2}\right] \tag{11}$$

Substituting this expression into the equation of motion, eqn (1), one obtains the following governing integro-differential equation

$$IG^*\left[\frac{\partial^4 w(x, t)}{\partial x^4}\right] + P\frac{\partial^2 w(x, t)}{\partial x^4} + m\frac{\partial^2 w(x, t)}{\partial t^2} = 0 \tag{12}$$

or, in terms of the inverse operator, J^*

$$\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{P}{I}J^*\left[\frac{\partial^2 w(x, t)}{\partial x^2}\right] + \frac{m}{I}J^*\left[\frac{\partial^2 w(x, t)}{\partial t^2}\right] = 0. \tag{13}$$

For a linear viscoelastic material, using the explicit expression of J^* in eqn (8), the governing equation is written as

$$\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{P}{EI}\left[\frac{\partial^2 w(x, t)}{\partial x^2} + j(t)^*\frac{\partial^2 w(x, t)}{\partial x^2}\right] + \frac{m}{EI}\left[\frac{\partial^2 w(x, t)}{\partial t^2}\right.$$

$$\left. + j(t)^*\frac{\partial^2 w(x, t)}{\partial t^2}\right] = 0 \tag{14}$$

where star (*) denotes the convolution, i.e.

$$j(t)^* w(x,t) = \int_o^t j(t-\tau)\, w(x,\tau)\,\mathrm{d}\tau. \tag{15}$$

3. EXACT SOLUTION

(i) *The dynamic analysis*: The exact solution to the governing integro-differential equation, eqn (14), is obtained by means of Laplace transforms. To this end, we assume the lateral deflection, $w(x, t)$, in the form

$$w(x,t) = F(t)\sin\frac{n\pi x}{l},\ n = 1,2,\ldots \tag{16}$$

which satisfies the boundary conditions, eqns (2) identically. The initial conditions in eqns (2) become

$$F(0) = 0, \frac{\mathrm{d}F(0)}{\mathrm{d}t} = v_o. \tag{17}$$

Substituting eqns (16) and (17) into eqn. (14), one obtains the following integro-differential equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2} + \omega_0^2\left(1-\frac{P}{P_e}\right)F + j(t)^*\left[\frac{\mathrm{d}^2F}{\mathrm{d}t^2} - \omega_o^2\frac{P}{P_e}F\right] = 0 \tag{18}$$

where P_e and ω_0 denote the classical Euler buckling load and natural frequency of the column, respectively, defined by

$$P_e = \frac{n^2\pi^2 EI}{l^2},\ \omega_o = \frac{n^2\pi^2}{l^2}\left(\frac{EI}{m}\right)^{1/2}. \tag{19}$$

Using the initial conditions in eqn (17), the Laplace transform of eqn (18) is obtained as

$$\bar{F}(s) = v_0\left\{\omega_o^2[1+\bar{j}(s)]^{-1} + \left(s^2 - \frac{P}{P_e}\omega_o^2\right)\right\}^{-1}. \tag{20}$$

Since the kernel, $j(t-\tau)$, is known for a given viscoelastic material, the time dependent deflection function, $F(t)$, can be obtained by taking its inverse transform.

Now determine $F(t)$ for a simple linear viscoelastic body referred to as the "three-element model" material (see Fig. 2). For this material, the kernel, $j(t-\tau)$, takes the following explicit form

$$j(t-\tau) = (\lambda-\mu)\,\mathrm{e}^{-\mu(t-\tau)} \tag{21}$$

when the coefficients λ and μ are defined in terms of the spring stiffness, E_1 and E_2, and the viscosity, ν_2, of the dashpot by

$$\lambda = \frac{E_1+E_2}{\nu_2},\ \mu = \frac{E_2}{\nu_2}. \tag{22}$$

Taking the Laplace transform of eqn (21) and substituting into eqn (20) the transformed function, $\bar{F}(s)$, takes the following form for this special case

$$\bar{F}(s) = v_o(\lambda+s)\left[s^3 + \lambda s^2 + \omega_o^2\left(1-\frac{P}{P_e}\right)s + \omega_o^2\mu\left(1-\frac{\lambda}{\mu}\frac{P}{P_e}\right)\right]^{-1}. \tag{23}$$

since the denominator in eqn (23) is a cubic function of s, the inverse transform cannot be taken without determining the numerical values for the coefficients λ, μ and ω_o and load ratio, P/P_e.

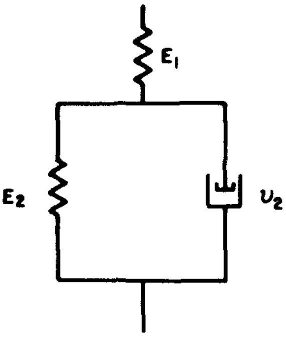

Fig. 2. Three element model for viscoelastic materials.

Now taking $E_1 = E_2 = 5.0 \times 10^6$ psi, and $\nu_2 = 1.72 \times 10^{13}$ psi sec., one obtains $\lambda = 2\mu = 5.787 \times 10^{-7}$/sec. Let the initial velocity at midspan be $v_0 = 0.025$ in./sec., and the dimensions of the column be: the cross-sectional area $A = 25.0$ in.^{2}; the half-depth of the cross-section $h = 2.887$ in.; the width of the cross-section $b = 4.330$ in.; and the column length $l = 200$ in. Therefore, the natural frequency is obtained as $\omega_0 = 48$/sec. Using the above data in eqn (23), the time-dependent deflection function, $F(t)$, is obtained for *several load levels*, and the time-dependent nondimensional amplitude $A(t)$ is plotted in Fig. 3. As can be seen from Fig. 3, if the applied load, P, is greater than a certain critical value, referred to as the "safe load limit", P_∞, in [11], (which is equal to $P_e/2$ for this special example), the amplitude of vibration decreases to a minimum at a certain time, and then starts to grow and becomes infinitely large as time goes to infinity. On the other hand, for $P \leq P_\infty = P_e/2$ the amplitude continues to decrease and finally goes to zero as time approaches infinity. We conclude, therefore, that the column is stable provided $P \leq P_\infty = P_e/2$; for $P > P_\infty = P_e/2$, it is unstable.

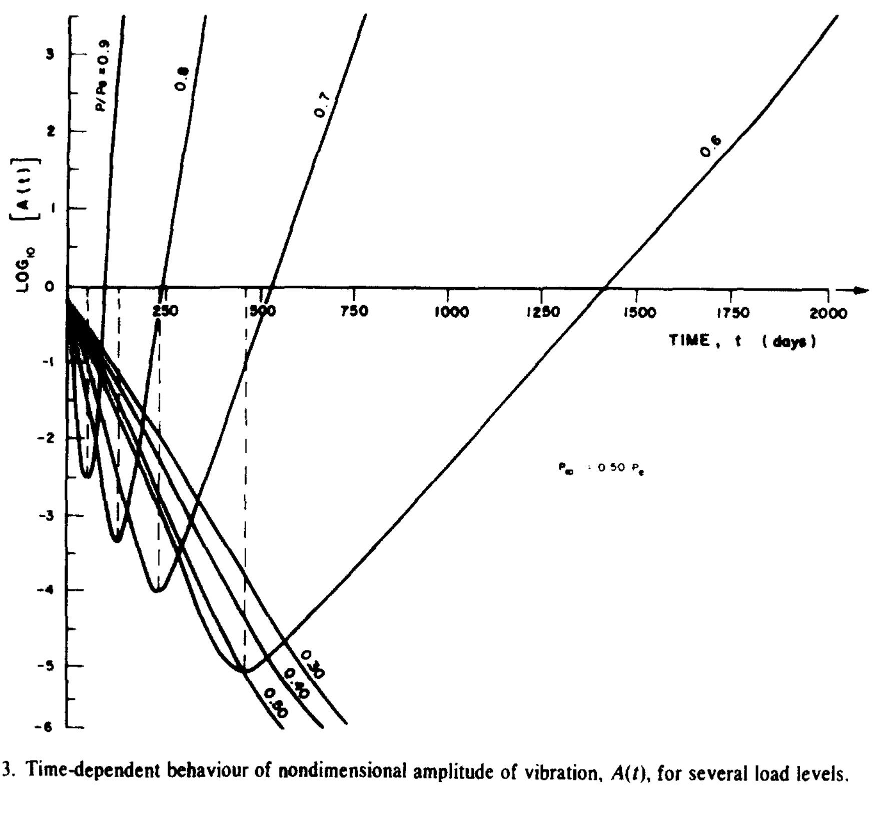

Fig. 3. Time-dependent behaviour of nondimensional amplitude of vibration, $A(t)$, for several load levels.

(ii) *Quasi-static solution.* A quasi-static solution may be derived by assuming that the inertia force term in eqn (14) is negligibly small. The integro-differential equation, eqn (14), then becomes

$$\frac{\partial^4 w(x,t)}{\partial x^4}+\frac{P}{EI}\left[\frac{\partial^2 w(x,t)}{\partial x^2}+j(t)*\frac{\partial^2 w(w,t)}{\partial x^2}\right]=0. \tag{24}$$

Using the expression for $w(x, t)$ in eqn (16), eqn (24) takes the form

$$F(t)-\frac{P}{P_e}\left(1-\frac{P}{P_e}\right)^{-1}\int_o^t j(t-\tau)\,F(\tau)\,\mathrm{d}\tau=0. \tag{25}$$

Since eqn (25) is a Volterra type homogeneous integral equation, it has only a trivial solution, i.e. $F(t)=0$. This erroneously implies that there is no buckling load lower than P_e.

4. APPROXIMATE SOLUTION

An approximate solution to the governing integro-differential equation, eqn (14), is derived by means of an approximation technique presented in[1]. This technique is based on the assumption that the Laplace transform of $\phi(t)$ exists, and its derivative is a slowly varying function of $\log_{10}t$ so that $\phi(t)$ and its transform, $\bar{\phi}(s)$, may be related by

$$\phi(t)=[s\bar{\phi}(s)]\Big|_{s=(0.5/t)} \tag{26}$$

with the inverse

$$s\bar{\phi}(s)\simeq[\phi(t)]\Big|_{t=(0.5/s)}. \tag{27}$$

Now taking the Laplace transform with respect to t of eqns (7) and (8), and using the convolution-product rule, on finds

$$\begin{aligned}\bar{\sigma}(x,s)&=E\{\bar{\epsilon}(x,s)+\bar{g}(s)\,\bar{\epsilon}(x,s)\}\\ \bar{\epsilon}(x,s)&=\frac{1}{E}\{\bar{\sigma}(x,s)+\bar{j}(s)\,\bar{\sigma}(x,s)\}.\end{aligned} \tag{28}$$

Define two new functions as

$$\psi(t)=\int_o^t j(t-\tau)\,\mathrm{d}\tau,\ \xi(t)=\int_o^t g(t-\tau)\,\mathrm{d}\tau \tag{29}$$

and take the Laplace transform of eqn (29)

$$s\bar{\psi}(s)=\bar{j}(s),\ s\bar{\xi}(s)=\bar{g}(s). \tag{30}$$

Substituting these definitions for $\bar{g}(s)$ and $\bar{j}(s)$ into eqns (28), and multiplying by s, one obtains

$$\begin{aligned}s\bar{\sigma}(x,s)&=E\{s\bar{\epsilon}(x,s)+s\bar{\xi}(s)s\bar{\epsilon}(x,s)\}\\ s\bar{\epsilon}(x,s)&=\frac{1}{E}\{s\bar{\sigma}(x,s)+s\,\bar{\psi}(s)\,s\bar{\sigma}(x,s)\}.\end{aligned} \tag{31}$$

Next, assume the approximation, eqn (27), is valid for $\sigma(x,t)$, $\epsilon(x,t)$, $\psi(t)$ and $\xi(t)$, in which case the approximate stress-strain relations are obtained as

$$\sigma_o(x, t) = E\{1 + \xi(t)\}\, \epsilon_o(x, t)$$
$$\epsilon_o(x, t) = \frac{1}{E}\{1 + \psi(t)\}\sigma_o(x, t) \tag{32}$$

where σ_o and ϵ_o denote the approximate stress and strain, respectively.

(i) *The dynamic analysis.* The use of these approximate stress–strain relations in the equation of motion leads to the following approximate governing differential equation for the stability of a column made of a Kelvin material

$$\frac{\partial^4 w_o(x, t)}{\partial x^4} + k^2(t)\frac{\partial^2 w_o(x, t)}{\partial x^2} + \alpha^2(t)\frac{\partial^2 \omega_o(x, t)}{\partial t^2} = 0 \tag{33}$$

where $w_o(x, t)$ denotes the approximate lateral deflection, and the coefficients, $k(t)$ and $\alpha(t)$, are defined by

$$k^2(t) = \frac{P}{EI}[1 + \Psi(t)],\ \alpha^2(t) = \frac{m}{EI}[1 + \Psi(t)]. \tag{34}$$

Assuming the approximate lateral deflection, $w_o(x, t)$, in the form

$$w_o(x, t) = \bar{F}_o(t)\sin\frac{n\pi x}{l} \tag{35}$$

the differential equation (33), becomes

$$\frac{d^2\bar{F}_o}{dt^2} + \omega_o^2[1 + \xi(t)]\bar{F}_o = 0. \tag{36}$$

For the three-element model material, $\xi(t)$ is given by

$$\xi(t) = -\frac{\lambda - u}{\lambda}(1 - e^{-\lambda t}). \tag{37}$$

Using eqn (37) and the following transformations

$$\tau = -\frac{\lambda t}{2},\ u = e^{\tau},\ F_o = \frac{\bar{F}_o}{l} \tag{38}$$

eqn (36) reduces to the following Bessel differential equation in u

$$u^2\frac{d^2F_o}{du^2} + u\frac{dF_o}{du} + (\gamma^2u^2 - \beta)F_o = 0 \tag{39}$$

where

$$\gamma^2 = \frac{\lambda - \mu}{\lambda}\frac{4\omega_o^2}{\lambda^2},\ \beta = \frac{4\omega_o^2}{\lambda^2}\frac{P - P_\infty}{P_e},\ P = P_e\frac{u}{\lambda}. \tag{40}$$

When $P > P_\infty$, $\beta = \nu^2 > 0$ then the solution to eqn (39) is obtained in the form

$$F_o(t) = v_o[A(o)\dot{B}(o) - B(o)\dot{A}(o)]^{-1}[A(o)B(t) - B(o)A(t)] \tag{41}$$

where

$$A(t) = \sum_{k=o}^{\infty}\frac{(-1)^k\gamma^{2k}}{2^{2k+\nu}k!\,\Gamma(\nu + k + 1)}e^{-(\lambda/2)(2k+\nu)t}$$

$$B(t) = \sum_{k=o}^{\infty} \frac{(-1)^k \gamma^{2k}}{2^{2k-\nu} k! \Gamma(-\nu+k+1)} e^{-(\lambda/2)(2k-\nu)t}$$

$$A(o) = \sum_{k=o}^{\infty} \frac{(-1)^k \gamma^{2k}}{2^{2k+\nu} k! \Gamma(\nu+k+1)}$$

$$B(o) = \sum_{k=o}^{\infty} \frac{(-1)^k \gamma^{2k}}{2^{2k-\nu} k! \Gamma(-\nu+k+1)}$$

$$\dot{A}(o) = -\frac{\lambda\nu}{2} A(o) - \lambda \sum_{k=o}^{\infty} \frac{(-1)^k \gamma^{2k}}{2^{2k+\nu}(k-1)! \Gamma(\nu+k+1)}$$

$$\dot{B}(o) = \frac{\lambda\nu}{2} B(o) - \lambda \sum_{k=o}^{\infty} \frac{(-1)^k \gamma^{2k}}{2^{2k-\nu}(k-1)! \Gamma(-\nu+k+1)}. \quad (42)$$

Consider the exponential terms in the approximate time-dependent deflection function, $F_o(t)$, in eqn (41). Since λ and ν are real and positive for $P_\infty < P < P_e$, the function $B(t)$ rapidly approaches infinity while $A(t)$ decays to zero, as $t \to \infty$, which implies that the column will buckle in a very short time, without vibrating, if it is disturbed by a lateral disturbance. The time-dependent function, $F_o(t)$, is plotted in Fig. 4 for several load levels, using the same column which was used in the "exact" solution. Figure 4 indicates that the deflection, $F_o(t)$, takes on very large values after only a very short time, i.e. the column will buckle almost instantly after load application. Comparing these approximate results with those obtained from the "exact" solution, we note that agreement between the two sets of results is very poor for load levels $P_\infty < P < P_e$. For example, for the load ratio, $P/P_e = 0.7$, the two solutions take the same values after 0.7 sec. and 750 days, respectively.

On the other hand, since the coefficient ν is imaginary for $P < P_\infty$, the time-dependent function, $F_o(t)$, includes only exponential terms with negative powers and periodic functions. Thus for such load levels the column will vibrate with the time-dependent amplitude decaying to zero as $t \to \infty$. We conclude, therefore, that the column is stable for $P < P_\infty$.

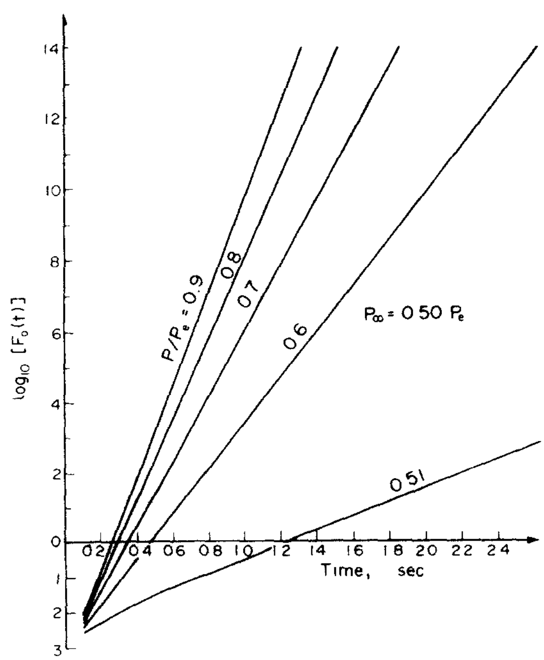

Fig. 4. Time-dependent behaviour of approximate nondimensional deflection, $F_o(t)$, for several load levels.

(ii) *Quasi-static solution*

A quasi-static solution may also be derived for this approximate case. Neglecting the inertia term in eqn (33), the governing equation is obtained as

$$\frac{\partial^4 w_o(x,t)}{\partial x^4} + k^2(t)\frac{\partial^2 w_o(x,t)}{\partial x^2} = 0. \tag{43}$$

Using the expression for $w_o(x, t)$ in eqn (35), we obtain

$$\left[\left(\frac{n\pi}{l}\right)^2 - k^2(t)\right]F_o(t) = 0. \tag{44}$$

For a non-zero solution, the coefficient of $F_o(t)$ should vanish which yields a continuous eigenvalue spectrum for the critical load, P_{cr}, [11], i.e.

$$P_{cr} = P_e[1 + \psi(t)]^{-1}. \tag{45}$$

In the case of a three element model column, the critical load, P_{cr}, and the corresponding finite critical time, t_{cr} for a given load $P_\infty < P < P_e$ are obtained, respectively as [11]

$$P_{cr} = P_e\left[1 + \frac{\lambda - \mu}{\mu}(1 - e^{-\mu t})\right]^{-1} \tag{46}$$

and

$$t_{cr} = -\frac{1}{\mu}\ln\left[1 - \frac{\mu}{\lambda - \mu}\left(\frac{P_e}{P} - 1\right)\right]. \tag{47}$$

The values of critical times are tabulated in Table 1 for several load ratios for a column made of a three-element model material.

When we compare these results for a continuous spectrum of eigenvalues, $P_\infty \le P \le P_e$, and corresponding "finite critical" times with those obtained from an "exact" solution, we must conclude that the approximate quasi-static solution and associated solution techniques and assumptions, used also in [11], lead to erroneous results.

CONCLUSIONS

The "exact" solution to the integro-differential equation defining the behaviour of a viscoelastic perfect column subjected to an axial load P and a lateral disturbance, shows that such a column will vibrate with a time-dependent amplitude as a result of the initial disturbance. If the applied load, P, is greater than the safe load limit, P_∞, the amplitude of vibration decreases to a minimum at a certain time, and then starts to grow and become infinitely large as $t \to \infty$ (see Fig. 3). On the other hand, for $P \le P_\infty$, the amplitude continues to decrease and decays to zero as $t \to \infty$. We conclude, therefore, that the column is stable provided $P \le P_\infty$: for $P > P_\infty$, it is unstable.

Table 1. Approximate critical times for several load levels

Load P/P_e	Ratios Critical Time, t_{cr} (days)
0.90	4.7
0.80	11.2
0.70	22.4
0.60	44.0
0.51	129.0
0.50	

It is noteworthy that if the method of separation of variables is applied to the quasi-static case, the governing integro-differential equation leads to a homogeneous Volterra integral equation which has only a trivial solution. The quasi-static solution indicates, therefore, that there is no buckling load lower than P_e. This shows that the quasi-static approach yields incorrect results for the stability of perfect columns made of linearly viscoelastic materials.

On the other hand, an approximate dynamic analysis shows that the lateral deflection of the column grows very rapidly with time, as soon as the column is disturbed and approaches infinity as $t \to \infty$, if $P > P_\infty$, (see Fig. 4). For $P \leq P_\infty$, the column will vibrate with a time-dependent amplitude as a result of such an initial disturbance. The amplitude decreases with time and finally decays to zero as $t \to \infty$.

It should be noted that for $P_\infty < P < P_e$, the deflections obtained from the approximate solution become very large, in a very short time, compared with the values of the deflection obtained from the "exact solution". For example, for a load ratio $P/P_e = 0.7$, the approximate deflection takes a certain value after 0.7 sec., while the "exact" solution indicates a deflection of similar magnitude only after 750 days.

The approximate solution for the quasi-static case also yields a continuous eigenvalue spectrum for load levels $P_\infty < P < P_e$, with associated "finite critical times" (see Table 1). Only for the case of $P = P_\infty$, does this critical time go to infinity, while for $P < P_\infty$, the approximate solution indicates no eigenvalues.

We conclude, therefore, that agreement between the results from the "exact" and "approximate" solutions in the dynamic analysis is poor for load levels $P > P_\infty$. We, also conclude, that the quasi-static approach does not yield correct results for either the "exact" or approximate solution.

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